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Young's modulus of porous brittle solids

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A new equation $E = E_0 (1 - aP)^n$ where E and E_0 are the Young's moduli at porosity, P, and zero, respectively, a and n are material constants, has been derived semi-empirically for describing the porosity dependence of Young's modulus of brittle solids. The equation satisfies quite well the exact theoretical solution for the values of Young's moduli at different porosities for model systems with ideal and non-ideal packing geometry. The equation shows excellent agreement with the data on α- and β-alumina over a wide range of porosity. Unlike the existing porosity–elastic modulus equations, the proposed equation satisfies the boundary conditions and is inherently capable of treating isometric closed pores as well as non-isometric interconnected pores. The parameters a and n provide information about the packing geometry and pore structure of the material.

1. Introduction

The relation

$$E = E_0 \exp(-bP) \quad (1)$$

where E and E_0 are the elastic moduli at porosity P and zero, respectively, and b, the material constant, proposed by Spriggs [1] has widely been used to predict the elastic modulus of porous brittle solids [2–5]. Equation 1 is purely empirical in nature and not based on theory. It has been criticised [6] because it fails to satisfy the boundary condition that E equals zero for $P = 1$. Wang [7] has theoretically derived a relation between porosity and Young's modulus for materials composed of uniform-sized spherical particles packed in cubic array and represented the exact solution graphically. He has shown that Equation 1 approximately agrees with the exact solution over a narrow range of porosity only at the lower end. The deviation becomes greater for $P \geqslant 0.2$. He concluded that for correlating data over a wider range of porosity the value of b in Equation 1 cannot be treated as constant and the exponent cannot be considered a linear function of porosity. Therefore, he proposed an approximate solution with a quadratic exponent:

$$E = E_0 \exp[-(bP + cP^2)] \quad (2)$$

which agreed with the exact solution over $P \leqslant 0.38$ well within $\pm 2\%$ accuracy. For still higher porosity he suggested a relation with polynomial exponent:

$$E = E_0 \exp[-(bP + cP^2 + dP^3 + \cdots)] \quad (3)$$

where b, c, d, ... are material constants. Even Equations 2 and 3 are not free from the limitations mentioned earlier, and an infinite number of terms has to be added to the exponent to obtain E equal to zero at $P \leqslant 1$. Also for an infinite number of terms the material constants b, c, d etc., lose their physical significance and practical use of such an equation becomes cumbersome. With a view to finding a solution of the problem, a new equation has been derived semi-empirically and compared with the exact solution of Wang [7] over the entire porosity range. The applicability of the equation to the data on α- and β-alumina given by Wang [7] and others [2, 8, 9] is also reported.

2. Theoretical derivations

Fig. 1 depicts a porous body of average porosity P having a constant area of cross-section A and length l, when the body is subjected to a load W in the x direction, the elongation $\Delta\delta$ of an element $A\,dx$ is given by:

$$\Delta\delta = (W/A_x)/E_0\,dx \quad (4)$$

here A_x is the solid area of cross-section available in element dx.

The total elongation δ of the body is then given by:

$$\delta = \Sigma\Delta\delta = \int_0^l W/E_0 A_x\,dx \quad (5)$$

If E is the apparent Young's modulus then

$$E = (W/A)/(\delta/l) \quad (6)$$

Combining Equation 5 and 6, the relation

$$\frac{1}{E} = \frac{1}{E_0 l}\int_0^l \frac{A}{A_x}\,dx \quad (7)$$

is obtained.

Now, if P_x is the average porosity of the element $A dx$, then

$$\frac{A_x}{A} = 1 - P_x$$

and Equation 7, therefore, reduces to

$$\frac{1}{E} = \frac{1}{E_0 l}\int_0^l dx/1 - P_x \quad (8)$$

The integral on the right-hand side can only be evaluated if P_x is expressed in terms of x. This is possible if a certain idealized system is assumed for the pore-distribution in the body. However, for a real system, Equation 8 can be written as

$$E = E_0 f(1 - P) \quad (9)$$

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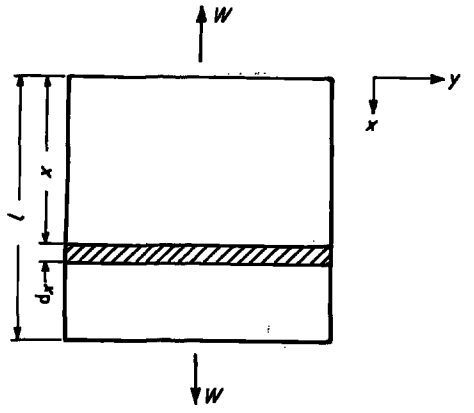

Figure 1 Porous body in tension: a schematic view.

where $f(1 - P)$ is a function of average porosity P. The function $f(1 - P)$ must be of such form that Equation 9 satisfies the boundary conditions

$$E = E_0 \quad \text{at} \quad P = 0$$

and

$$E = 0 \quad \text{at} \quad P \leqslant 1$$

The simplest mathematical function which satisfies such boundary conditions is a power function of the form $(1 - aP)^n$ where a and n are material constants. Hence Equation 9 can be expressed as

$$E = E_0 (1 - aP)^n \tag{10}$$

Since Equation 10 satisfies the boundary condition $E = 0$ at $P \leqslant 1$ the material constant a can be defined as $a = 1/P_{crit}$, where P_{crit} is the critical porosity at which elastic modulus becomes zero. Schiller [10] has defined P_{crit} in a similar way while investigating strength properties of gypsum. Knudsen [11] has studied contact area as a function of bulk density and theoretically derived the values of P_{crit} for polycrystalline materials composed of uniform spherical particles arranged in cubic, orthorhombic and rhombohedral array as 0.476, 0.397 and 0.26, respectively. The corresponding values of a become 2.10, 2.52 and 3.85. The minimum value of a is 1 corresponding to the maximum value of $P_{crit} = 1$. The value of a, therefore, lies in the range $1 \leqslant a \leqslant 3.85$. Thus a may be defined as the packing geometry factor.

Fig. 2 shows the theoretical curves for Young's modulus of porous materials given by Wang [7] along with Equation 10 fitted to the same curve. Spriggs' correlation as well as Wang's approximate solution is also plotted in Fig. 2. It is evident from Fig. 2 that Equation 10 closely approximates the theoretical curves over almost the entire range of porosity, the accuracy being well within $\pm 2\%$ up to the porosity range of 0.4726. The values of (a, n) are equal to (1.8, 0.651) (1.72, 1.347) and (1.641, 2.181) for curves 1, 2 and 3, respectively, given by Wang [7].

3. Application to α- and β-alumina

The porosity–Young's modulus data on α-alumina reported by Wang [7] are shown in Fig. 3 along with the theoretical solution for the non-ideal case (combined shear and hinge effect) as well as the equation proposed by Wang [7] and the present authors. Out of the 40 reported values of Young's modulus at different porosities, all except three could be read because of closeness of the data. The theoretical plot was made taking the E_0 value to be equal to the reported Reuss–Hill average [12] 402.85 GPa. As suggested by Wang [7], Equation 2 was fitted to the data only up to 38%

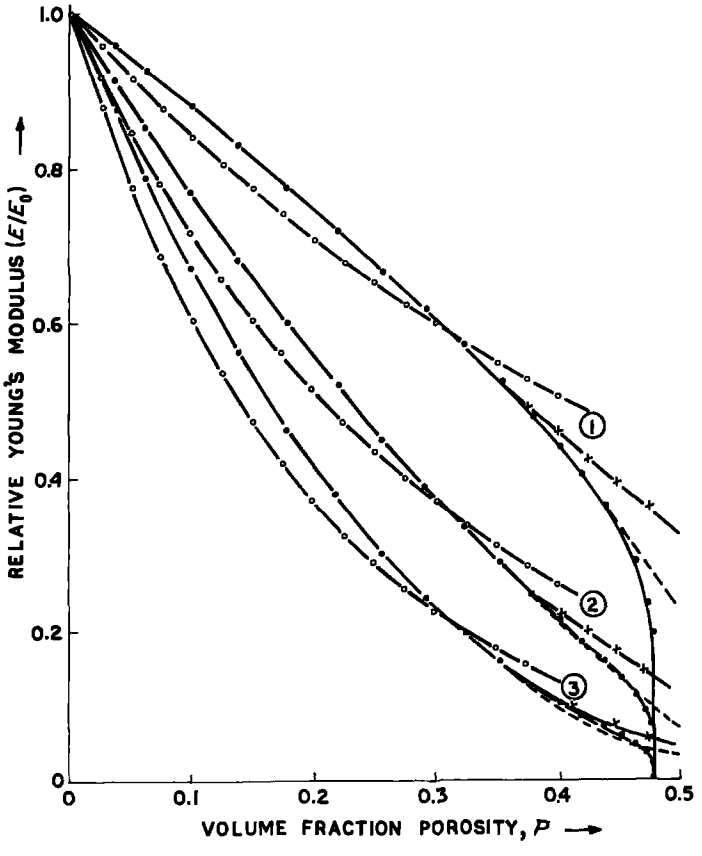

Figure 2 Young's modulus as a function of porosity. The exact solution (●), Spriggs' correlation (○), Wang's correlation (×) and the proposed equation (———) are shown in the plot. (1) Ideal case, (2) non-ideal case (shear effect), (3) non-ideal case (combined shear and hinge effect).

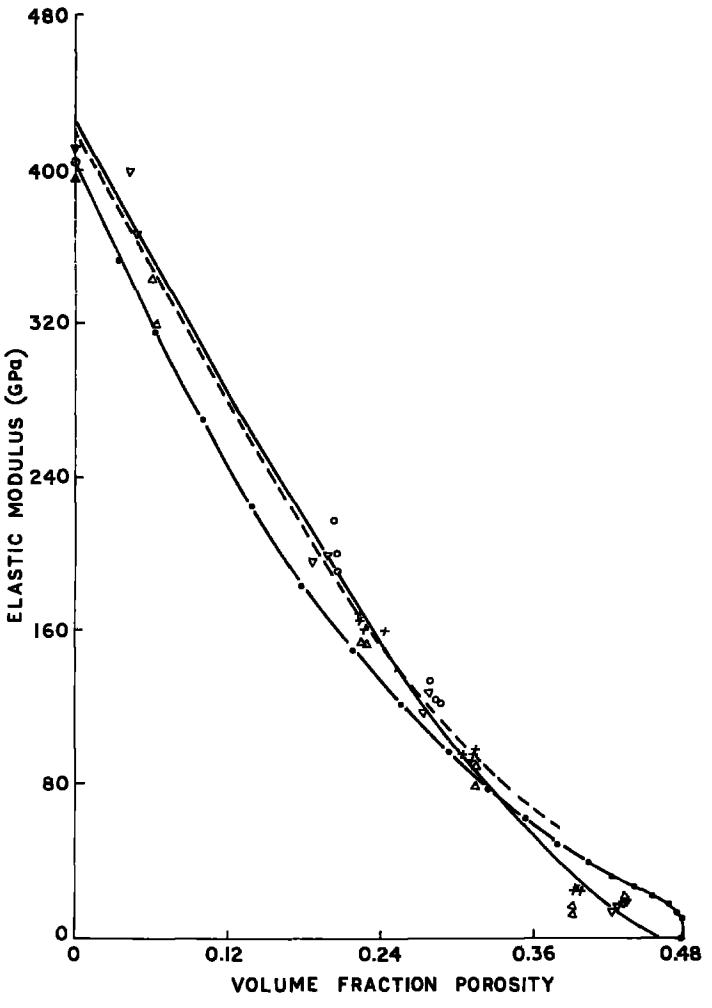

Figure 3 Young's modulus plotted against porosity of α-alumina. A total of 37 data points as reported by Wang [7] is shown in the plot. (▼) Voigt bound, (▲) Reuss bound, (○) Hill average. (—·—) Theoretical equation, (– –) Wang's equation, (——) proposed equation.

porosity following the least square method taking a log–log relation between the factors involved, thus giving the values of E_0 = 403.98 GPa, b = 1.77 and c = 9.35 instead of 401 GPa, 1.46 and 9.82, respectively, reported by him. Since the change of variables to log scale will affect the data scatter differently at different points on the curve and also the positive and negative errors differently at the same point on the curve, the equation was fitted following the iterative least square analysis given by Lewis [13]. A computer program was run and the values of the parameters E_0, b and c obtained are given in Table I along with the standard error of estimate.

Based on the iterative least square analysis of Lewis [13], a method for fitting the proposed equation 10 was evolved and is given in the Appendix. For the data fitted over the entire porosity range the values of E_0, a and n are given in Table I along with the standard error of estimate. Computer programs were also run for Spriggs' equation, the linear equation $E = E_0(1 - hP)$ and Hasselman's equation [6]

$$E = E_0\left[1 + \frac{AP}{1 - (A + 1)P}\right]$$

and the values of all the parameters obtained are also given in Table I.

As can be seen from Table I, the proposed equation best describes the data having minimum standard error of estimate. The value of E_0 = 425.95 GPa obtained from this equation also agrees quite well with the Reuss–Hill average, the deviation being only 5.7%. It may be mentioned that though Wang [7] has reported E_0 = 401 GPa on the basis of linear least square method, the value of E_0 increases to 418.68 GPa on evaluation by iterative least square method. The value of a = 2.76 indicates that E becomes zero at a porosity of 45.96% which is very close to that for cubic packing, i.e. 47.64%. Thus this confirms Wang's observation regarding the packing geometry of the material studied, although his equation fails to provide any specific information regarding this. It may also be mentioned that although the proposed equation has been fitted over the entire range of porosity (0 to 43.0%), it still provides a standard error of estimate lower than that obtained for Wang's equation.

Fig. 4 shows the porosity–Young's modulus data on α-alumina given by Knudsen [2] along with the plots of Wang's equation as well as the proposed equation fitted to the same curve. The data comprises 71 values of Young's modulus over the porosity range (1 to 41%) reported by various investigators [14–19] of the material prepared by different fabrication processes, namely hot pressing, cold pressing, extrusion and casting. Spriggs' equation, linear equation and Hasselman's equation were fitted to the data as before. The values of all the parameters of the equation obtained are given in Table I. Of all the equations fitted over the entire range of porosity the proposed equation provides the best fit, having minimum standard error of estimate. Wang's equation provides a marginally lower standard error of estimate since it

TABLE I Test data parameters of α- and β-alumina obtained from regression analysis

Source of test data	(1) $E = E_0 [\exp(-bP)]$	(2) $E = E_0 \exp[-(bP + cP^2)]$ (fitted over the porosity range $0 \leqslant P \leqslant 0.38$ only)	(3) $E = E_0 (1 - hP)$	(4) $E = E_0\left[1 + \frac{AP}{1 - (A + 1)P}\right]$	(5) $E = E_0 (1 - aP)^n$
Wang†	E_0 = 502.55 GPa b = 5.431 standard error 26.57	E_0 = 418.68 GPa b = 2.405 c = 7.376 standard error 13.72	E_0 = 384.93 GPa h = 2.372 standard error 17.89	yields unrealistic E_0 value	E_0 = 425.95 GPa a = 2.176 n = 1.365 standard error 12.24
Knudsen†	E_0 = 407.11 GPa b = 3.861 standard error 15.16	E_0 = 407.82 GPa b = 3.955 c = −0.489 standard error 10.93	E_0 = 385.80 GPa h = 2.319 standard error 21.27	E_0 = 425.66 GPa A = −5.611 standard error 14.80	E_0 = 402.28 a = 1.0 n = 3.371 standard error 11.81
Coble and Kingery†	E_0 = 428.92 GPa b = 2.969 standard error 6.80	Not fitted because of insufficient data	E_0 = 357.85 GPa h = 1.548 standard error 10.62	E_0 = 498.40 GPa A = −4.435 standard error 12.68	E_0 = 391.69 GPa a = 1 n = 2.143 standard error 1.77
Evans *et al**	E_0 = 215.0 GPa b = 5.037 standard error 4.18	E_0 = 213.59 GPa b = 4.840 c = 0.575 standard error 4.78	E_0 = 192.32 GPa h = 2.353 standard error 11.1	E_0 = 118.06 GPa A = −4.341 standard error 65.55	E_0 = 208.22 GPa a = 1 n = 4.120 standard error 4.55

(1), (2) and (5) ILS; (3) and (4) LLS, †α-alumina, *β-alumina.

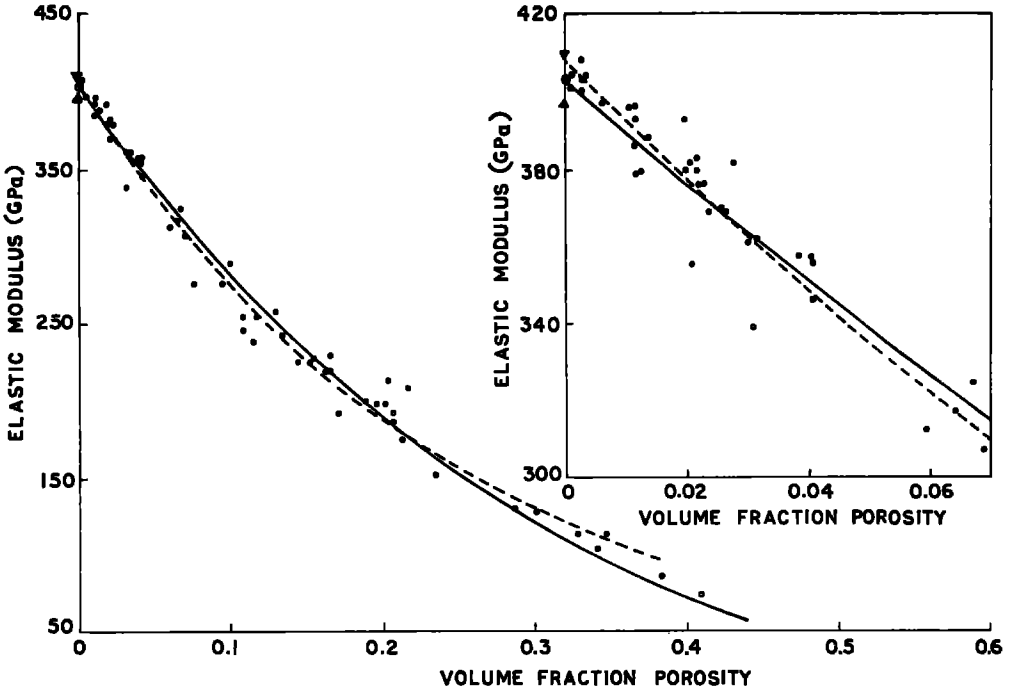

Figure 4 Young's modulus plotted against porosity of α-alumina. A total of 71 data points as compiled by Knudsen [2] is shown in the plot. (▼) Voigt bound, (▲) Reuss bound, (O) Hill average. (---) Wang's equation, (——) proposed equation.

has been fitted only up to 38% porosity. The value of $E_0 = 402.28$ GPa obtained from the proposed equation is within 0.1% of the Reuss–Hill average. The value of $a = 1$ indicates that Young's modulus of the material becomes zero only at 100% porosity. It is apparent from Figs 3 and 4 that above a porosity of about 20%, the values of Young's modulus reported by Wang [7] decrease at a higher rate with increasing porosity than those compiled by Knudsen [2], the value being almost one-fourth of that in the latter case in the porosity range 36 to 38%. Wang [7] has attributed this to the transition of the pore structure from a disconnected to an interconnected one, above 20% porosity. In an ordered packing such as cubic packing, the contact area is largely reduced at higher porosity, consequently leading to much lower values of Young's modulus. The high values of Young's modulus at high porosity in the latter case, therefore, can be explained in terms of the pore structure being less interconnected due to the randomness of packing, which leads to the value of $a = 1$ in this case.

The porosity–Young's modulus data on α- and β-alumina reported by Coble and Kingery [8] and Evans *et al.* [9], respectively, are shown in Fig. 5 along with the regression lines fitted to the same curves. Coble and Kingery [8] derived the relation

$$E = E_0(1 - 1.91P + 0.91P^2)$$

from Mackenzie's expression [20] on the basis of the boundary condition $E = 0$ at $P = 1$. This equation has also been fitted to their data only and shown in Fig 5. The equation yielded a value of $E_0 = 368.8$ GPa (4% lower than the value quoted by them, since the data have been taken from their plot) with the standard error of estimate 9.69. Here again, the proposed equation provided the best fit with the value of $E_0 = 391.69$ GPa (Table I) which is only 2.8% lower than the Reuss–Hill average. The value of $a = 1$ also agrees with the assumption of Coble and Kingery for boundary conditions. It may be noted that even though the data cover a porosity range of 50.5% Spriggs' equation provides a value of $E_0 = 428.92$ GPa which is 6.5% higher than the Reuss–Hill average with a standard error of estimate less than that of Coble and Kingery [8].

For data on β-alumina over porosity range 2 to 37%, the proposed equation provides a value of $E_0 = 208.22$ GPa with a standard error of estimate slightly higher than that of Spriggs' equation (Table I). Spriggs' equation provides the value of $E_0 = 215.0$ GPa. It may, however, be mentioned that the value reported by Evans *et al.* [9] is 205 GPa from Spriggs' equation.

While investigating his own data on α-alumina, Wang [7] showed that narrowing down the porosity range to $P \leqslant 0.2$, Spriggs' equation provides a good fit yielding the value of $E_0 = 425$ GPa, very close to the value of E_0 established in the literature. Therefore, he concluded that this equation is for both open and closed pores and for relatively small porosity ranges; his modification is for both open and closed pores over a wider porosity range. As can be seen from Table I, both in the case of Knudsen's and Evans *et al*'s [2, 9] data, Spriggs' equation has been fitted up to to a porosity of 41 and 37%, respectively, yielding values of E_0 which are almost identical with those given by Wang's [7] equation by fitting up to the porosity range of 38%. It is also to be noted that in the case of data on β-alumina [9] Spriggs' equation shows a better correlation having a standard error of estimate 4.18 as against 4.78 in the case of Wang's equation. Also the material constant c in Wang's equation has a negative value in the case of Knudsen's data, thereby losing its physical significance. This is in

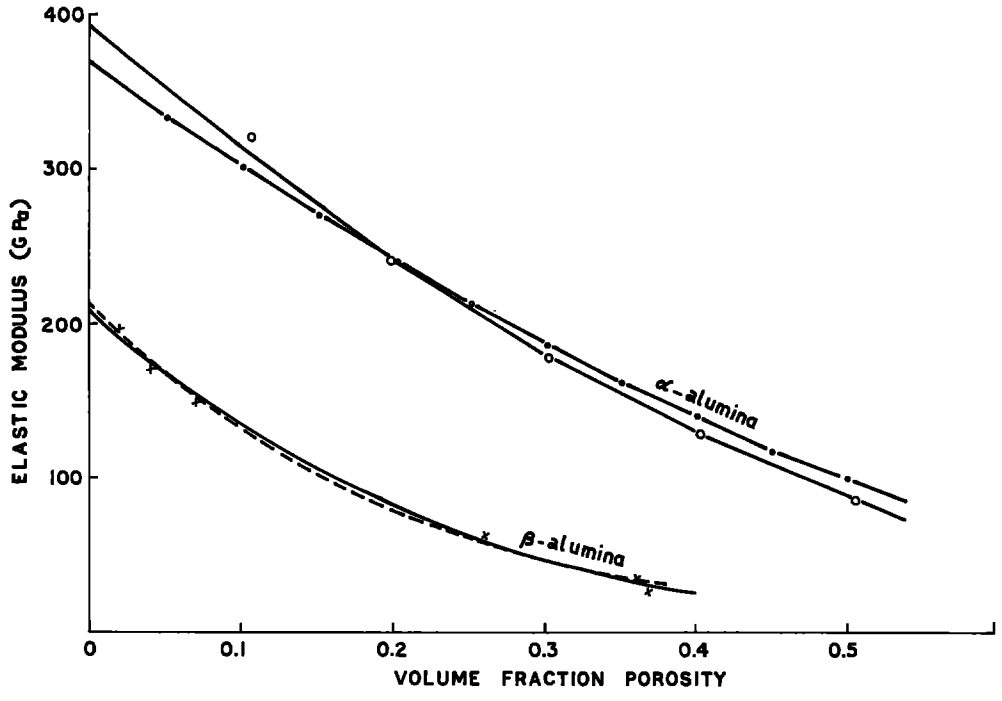

Figure 5 Young's modulus plotted against porosity of α- and β-alumina. A total of 5 data points of α-alumina as reported by Coble and Kingery (○) [8] and that of 6 data points of β-alumina as reported by Evans *et al.* (×) [9] are shown in the plot. (– – –) Wang's equation, (——) proposed equation, (–·–) Mackenzie's equation.

contradiction to Wang's suggestion that all the coefficients of the exponent are non-negative numbers. It may, therefore, be concluded that Wang's observations on the applicability of Spriggs' equation as well as that proposed by him, possibly hold good for ordered packing only. A reasonably good fit of Spriggs' equation with the data of Knudsen [2], Coble and Kingery [8] and Evans *et al.* [9] over the entire range of porosity under study suggests that the range of porosity over which an exponential equation can be correlated depends on factors such as packing geometry, pore structure, etc. This inherent limitation of exponential equation is due to the fact that the expression describing the physical phenomenon does not satisfy the boundary conditions, affecting thereby the values of the constants evaluated by fitting to experimental data [6].

As can be seen from Table I, for the last three sets of data, the value of $a = 1$ indicates similar packing in all three cases. However, the value of n differs in each case having a minimum value of 2.14 in case of Coble and Kingery [8] and a maximum value of 4.12 in case of Evans *et al.* [9]. Coble and Kingery [8] analysed their data by treating their material as a continuous solid phase with discontinuous isometric pores and obtained a good correlation with Mackenzie's equation for a continuous solid phase with isolated spherical pores.

Analysis of the porosity–elastic modulus data on sintered glass with spherical pores [21] yielded values of $n = 2.15$ and 2.05 for Young's modulus and shear modulus, respectively, with $a = 1$ in both cases, suggesting that for random packing with closed spherical pores, the value of n will be approximately equal to 2. The value of $n = 2.14$ obtained from the data of Coble and Kingery [8], therefore, indicates that their material possibly contained almost spherical closed pores. This is in agreement with their own observation for the material studied. The value of n for β-alumina is 4.14. A study of the micrographs of the material given by Evans *et al.* [9] shows that the pores are very much irregular in shape and also interconnected. This possibly led to rapid decrease of Young's modulus with increasing porosity, thereby, giving a high value of n. Similar observations [22] have been made for gypsum which is known to have acicular grain structure with interconnected pores of highly irregular shape [10]. Thus, for Knudsen's data with $n = 3.37$ it can be concluded that the pores deviated from the spherical shape and were interconnected to a certain extent. The parameter n, therefore, is related to grain morphology and pore structure of the material. The lower the value of n, the more isometric and isolated is the pore phase and vice versa. For a specific packing geometry, the proposed equation is in agreement with the observations of Coble and Kingery [8] that minimum changes in elastic modulus will occur when the solid phase is continuous; the largest changes occur when the pore phase is continuous.

4. Conclusion

A new equation $E = E_0 (1 - aP)^n$ has been derived semi-empirically to describe the porosity dependence of Young's modulus. The material constant a in the equation may be defined as the "packing geometry factor", the value of which lies between 1 and 3.85. The other material constant, n, is dependent on grain morphology and pore geometry of the material. The present equation is capable of treating the transition of the pore structure from interconnected to isolated and has no limitation with regard to its applicability over any range of porosity.

The equation showed good agreement with the data

on α- and β-alumina yielding E_0 values very close to those reported in the literature. The values of n increased from 2.14 for closed spherical pores, to 4.12 for interconnected pores of irregular shape.

Appendix

An initial estimate of a, i.e. a_0 can be made from the plot E against P.

The initial estimates of E_0 and n_0 can then be made by the least square method assuming a log–log relationship between the factors involved.

The next best estimate of a_1, E_1 and n_1 is then obtained from

$$n_1 = n_0 + \delta n$$

$$a_1 = a_0 + \delta a$$

$$E_1 = E_0 + \delta E$$

where δn, δa and δE are obtained by solving the equations:

$$S_{13}\delta E + S_{23}\delta n + S_{33}\delta a = S_{43}$$

$$S_{11}\delta E + S_{12}\delta n + S_{13}\delta a = S_{41}$$

$$S_{12}\delta E + S_{22}\delta n + S_{23}\delta a = S_{42}$$

where

$$S_{11} = \Sigma B^2$$

$$S_{22} = \Sigma C^2$$

$$S_{33} = \Sigma D^2$$

$$S_{12} = \Sigma BC$$

$$S_{13} = \Sigma BD$$

$$S_{23} = \Sigma CD$$

$$S_{43} = \Sigma(E - A)D$$

$$S_{41} = \Sigma(E - A)B$$

$$S_{42} = \Sigma(E - A)C$$

and

$$A = E_0 \exp [n_0 \ln (1 - a_0 P)]$$

$$B = \exp [n_0 \ln (1 - a_0 P)]$$

$$C = E_0 \ln (1 - a_0 P) \exp [n_0 \ln (1 - a_0 P)]$$

$$D = E_0 [\exp \{n_0 \ln (1 - a_0 P)\}] \cdot (-n_0/1 - a_0 P)$$

A better estimate of a_2, E_2 and n_2 can then be obtained from a_1, E_1 and n_1 following the same procedure and the process can be continued until the desired degree of accuracy has been obtained.

The criteria

$$\left|\frac{\delta a_i}{a_i}\right| + \left|\frac{\delta n_i}{n_i}\right| + \left|\frac{\delta E_i}{E_i}\right| < 0.001$$

will provide a three-figure accuracy in the determined values.

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References

1. R. M. SPRIGGS, *J. Amer. Ceram. Soc.* **44** (1961) 628.
2. F. P. KNUDSEN, *ibid.* **45** (1962) 94.
3. I. SOROKA and P. J. SEREDA, *ibid.* **51** (1968) 337.
4. R. D. CARNAHAN, *ibid.* **51** (1968) 223.
5. D. F. PORTER, J. S. REED and DAVID LEWIS III, *ibid.* **60** (1977) 345.
6. D. P. H. HASSELMAN, *ibid.* **45** (1962) 452.
7. J. C. WANG, *J. Mater. Sci* **19** (1984) 801.
8. R. L. COBLE and W. D. KINGERY, *J. Amer. Ceram. Soc.* **39** (1956) 377.
9. J. R. G. EVANS, R. STEVENS and S. R. TAN, *Trans. J. Brit. Ceram. Soc.* **83** (1984) 43.
10. K. K. SCHILLER, in "Mechanical Properties of Non-Metallic Brittle Materials", edited by W. H. Walton (Butterworth, London, 1958) p. 35.
11. F. P. KNUDSEN, *J. Amer. Ceram. Soc.* **42** (1959) 376.
12. J. B. WACHTMAN Jr, W. E. TEFFT, D. G. LAM Jr and R. P. STINCHFIELD, *J. Res. Nat. Bur. Stand.* **64A** (1960) 213.
13. DAVID LWEIS III, *Ceram. Bull.* **57** (1978) 434.
14. W. H. DUCKWORTH, J. K. JOHNSTON, L. R. JACKSON and H. Z. SCHOFIELD, "Mechanical properties of Ceramic Bodies", Rand Report No. R-209, Battelle Memorial Institute, Columbus, Ohio (1950).
15. R. M. SPRIGGS and T. VASILOS, "Effect of grain size and porosity on transverse bend strength and elastic modulus of hot pressed alumina and magnesia", paper presented at the Sixty Third Annual Meeting, The American Ceramic Society, Toronto, Ontario, Canada, 24 April (1961).
16. S. M. LANG, "Properties of High-temperature ceramics and cermets – Elasticity and Density at Room Temperature", National Bureau of Standards Monograph no. 6 (NBS, Washington, 1960) p. 45.
17. H. Z. SCHOFIELD, J. F. LYNCH and W. H. DUCKWORTH, "Fundamental Studies of Ceramic Materials", Final Summary Report, Battelle Memorial Institute Report to office of Naval Research, Navy Dept. Contract No. N5 ORL - 111, 31 March (1949).
18. B. L. MAJUMDER, *Trans. Indian Ceram. Soc.* **11** (1952) 168.
19. D. H. CHUNG, "Elastic and Anelastic Properties of Fine Grained Polycrystalline Alumina at Elevated Temperatures", Bulletin of Research Depatment, Monthly Report No. 297, State University of New York College of Ceramics at Alfred University March (1961).
20. J. K. MACKENZIE, *Proc. Phys. Soc. (London)* **63B** (1950) 2.
21. D. P. H. HASSELMAN and R. M. FULRATH, *J. Amer. Ceram. Soc.* **47** (1964) 52.
22. K. K. PHANI, S. K. NIYOGI, A. K. MAITRA and M. ROY CHOUDHURY, *J. Mater. Sci.* **21** (1986) 4335.

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